EXTENSIONS OF HAAR MEASURE FOR COMPACT CONNECTED ABELIAN GROUPS

BY

GERALD L. ITZKOWITZ

(Communicated by Prof. H. FREUDENTHAL at the meeting of September 26, 1964)

1. *Introduction*

In 1950, KAKUTANI and OXTOBY [13] proved the existence of an extension of Haar measure on the circle to a countably additive translation and inversion invariant measure of character $2^{\mathfrak{c}}$ (for the definition of character see 6.1). It is well known that the cardinal of the circle is $2^{\aleph_0}=\mathfrak{c}$. Thus a natural question to ask is whether it is possible to extend Haar measure for an arbitrary compact Abelian group of cardinal $\mathfrak{n}>\aleph_0$ so that the new measure has character $2^{\mathfrak{n}}$. In this paper we show that the answer is in the affirmative if the group is connected. The author wishes to thank Professor K. A. Ross for the suggestion of this problem. This work will be used in partial fulfillment of the requirements for the Ph. D. degree at the University of Rochester.

The method of proof used was motivated by a paper of HULANICKI [10] and is in fact the natural extension of the methods employed by KAKUTANI and KODAIRA in a paper [12] that appeared simultaneously with the above quoted paper of KAKUTANI and OXTOBY. In their paper Kakutani and Kodaira showed that there was an extension of Haar measure on the circle of character $\mathfrak{c}$. Their method was to use the divisibility of the circle group T and of a large product group G of circles to get an algebraic embedding φ of T into G such that $\varphi(T)$ had Haar outer measure one in G.

In the present work we shall proceed similarly. Our method will be to algebraically embed a divisible group H into a large product group $G=\mathbf{P}_t H_t$ (where $H_t=H$ for all t) so that the resulting embedding has Haar outer measure one in G. We will then employ the notion of induced measure spaces introduced by DOOB [3] to finally prove the theorem.

As we remarked above, the method of proof was motivated by a paper of HULANICKI [10]. In this paper Hulanicki proved that there were dense subsets of full outer measure in products of separable measure spaces. He applied these results to show that each compact Abelian topological group G of cardinal $2^{2^{\mathfrak{n}}}(\mathfrak{n}\geqslant\aleph_0)$ has a dense subgroup of cardinal $\mathfrak{n}^{\aleph_0}$ of Haar outer measure one in G. He also showed that the method of Kakutani and Kodaira could be used to prove existence of an extension of Haar measure on the circle of character $2^{\mathfrak{c}}$.

In the first part of our paper we shall prove several generalizations of Hulanicki's product theorem. We shall then apply a special case of these results to prove that a compact Abelian topological group of cardinal $2^{2^{\mathfrak{n}}}$ contains a dense pseudocompact subgroup of cardinal $\mathfrak{n}^{\aleph_0}$.

The second part of our paper will be concerned with the problem of extension of Haar measure. To begin the proof of the existence of an extension of Haar measure we will use a special case of the Hulanicki type theorem.

2. *Some observations*

Let A be any set. Then $|A|$ denotes the cardinal of A; $\mathfrak{n}$ will always denote an infinite cardinal. If G is a topological group, then $\hat{G}$ will denote its character group. If X is a topological space, then $w(X)$ is the least cardinal of a basis of open sets for X.

It is a theorem of KAKUTANI [11] and PONTRYAGIN [15] that if G is a locally compact Abelian group then $w(G)=w(\hat{G})$. KAKUTANI [11] has also shown that if G is a discrete group then $|\hat{G}|=2^{|G|}$. These facts will allow us to avoid using the continuum hypothesis in the proof of our theorems.

We may observe that for G discrete $w(G)=|G|$. Thus if G is compact and if $w(G)=\mathfrak{n}$ it follows that $|\hat{G}|=\mathfrak{n}$ and $|G|=2^{\mathfrak{n}}$.

HARTMAN and HULANICKI [5], assuming the generalized continuum hypothesis, have shown that each compact Abelian group H satisfying $|H|\leqslant 2^{2^{\mathfrak{n}}}$ contains a dense subset of cardinal at most $\mathfrak{n}$. We shall complement this theorem in Theorem 5.6 of this paper. We shall be able to do away with the assumption of the generalized continuum hypothesis; however our conclusion will be slightly weaker, i.e., $\mathfrak{n}$ will be replaced by $\mathfrak{n}^{\aleph_0}$.

This theorem will be used below to prove an amusing generalization of the theorem stating that there are enough characters of a compact Abelian group to separate points.

Let G be any group; then G_d will denote the group G endowed with the discrete topology. It is an elementary theorem of COMFORT and ROSS [2] that if G is an Abelian group and if K is a subgroup of $(G_d)^\wedge$ then K is point-separating on G if and only if K is dense in the compact group $(G_d)^\wedge$.

2.1. Theorem.[1]) Suppose the continuum hypothesis holds. Let G be an infinite Abelian group satisfying $|G|=2^{\mathfrak{n}}$. Then there is a point separating group of characters K such that $|K|=\mathfrak{n}$. (K need not be a collection of continuous characters since G need not have a topology.)

Proof. Since $|G_d|=|G|=2^{\mathfrak{n}}$, Kakutani's theorem implies that the compact group $(G_d)^\wedge$ satisfies $|(G_d)^\wedge|=2^{2^{\mathfrak{n}}}$. Thus the theorem of Hartman and Hulanicki implies that there is a dense subset of $(G_d)^\wedge$ having cardinal

---

1) It was pointed out to the author by Professor Ross that Theorem 2.1 is essentially due to KAKUTANI [11]. However we give here a different proof.

$\mathfrak{n}$. Let K be the group generated by this set. Clearly $\mathfrak{n} \leqslant |K| \leqslant \mathfrak{n} \cdot \aleph_0 = \mathfrak{n}$ and K is point separating by the theorem of Comfort and Ross.

Remark. If G is a compact Abelian group satisfying $w(G) = \mathfrak{n}$ so that $|G| = 2^{\mathfrak{n}}$, then there are $\mathfrak{n}$ continuous characters and they separate points. Thus we have shown that this property is true for all Abelian groups provided we drop the requirement that the separating group must consist of continuous characters.

3. *A Separation Theorem*

3.1. Lemma. Let H be a compact Abelian group satisfying $w(H) \leqslant \mathfrak{n}$. Then H has at most $\mathfrak{n}^{\aleph_0}$ closed G_δ sets.

Proof. By the theorem of Kakutani and Pontryagin there are at most $\mathfrak{n}$ continuous characters in $\hat{H}$. Since rational combinations of characters separate points we may apply the Stone–Weierstrass theorem to conclude that there are at most $\mathfrak{n}^{\aleph_0}$ continuous functions. Since each zero set is given by a continuous function in H, i.e. each zero set is of the form $\{x \in H : f(x) = 0, f \text{ continuous}\}$, it follows that there are at most $\mathfrak{n}^{\aleph_0}$ zero sets in H. To complete the proof we need only observe that in a normal space every closed G_δ set is a zero set.

Remark. In a normal space X closed G_δ sets and zero sets are the same (Since each zero set Z is of the form $Z = \{x \in X : f(x) = 0, f \text{ continuous}\}$ it is immediate that

$$Z = \bigcap_{n=1}^{\infty} \{x \in X : f(x) < 1/n\}$$

so Z is a closed G_δ. Conversely, let F be a closed G_δ, so that $F = \bigcap_{n=1}^{\infty} U_n$ where each U_n is open. Since X is normal there is for each n a continuous function $0 \leqslant f_n \leqslant 1$ such that f_n is zero on F and one on the complement of U_n. Thus the function $f = \sum_{n=1}^{\infty} 2^{-n} f_n$ has zero set F). Thus since we are dealing with compact Abelian topological groups in our paper and these are always normal spaces we will use the two terms interchangeably.

Let A be any point set. Suppose that $|A| = 2^{\mathfrak{n}}$; then there is a $1-1$ correspondence of A with the compact product group $\{-1, 1\}^{\mathfrak{n}}$. By Lemma 3.1 this latter group has at most $\mathfrak{n}^{\aleph_0}$ closed G_δ sets. Furthermore, it is clear that there are at most $(\mathfrak{n}^{\aleph_0})^{\aleph_0} = \mathfrak{n}^{\aleph_0}$ distinct sequences of closed G_δ sets in this space.

3.2. Lemma. Let $\{t_i\}_{i=1}^{\infty}$ be any countable sequence of points in a completely regular space T; then there is a sequence $\{Z_i\}_{i=1}^{\infty}$ of pairwise disjoint zero sets satisfying $t_i \in Z_i$ for each i.

Proof. By induction.

Let $n \geqslant 0$ be an integer and assume that we have found disjoint zero sets $Z_k (1 \leqslant k \leqslant n)$ such that $t_k \in Z_k$ and $t_m \notin Z_k$ if $m > n$.

By complete regularity there is a zero set A_i containing t_{n+1} and disjoint from Z_i for each $i \leqslant n$. Similarly there are zero sets A_i containing t_{n+1} and disjoint from t_{i+1} for each $i > n+1$. Let $Z_{n+1} = \bigcap_{i=1}^{\infty} A_i$, then Z_{n+1} has the desired properties.

3.3. Theorem. Let T be any set such that $|T| = 2^{\mathfrak{n}}$; then there exists a family $\mathfrak{A}$ of sequences $\{B_i\}_{i=1}^{\infty}$ of pairwise disjoint subsets of T such that

(i) $|\mathfrak{A}| \leqslant \mathfrak{n}^{\aleph_0}$

(ii) for any distinct sequence $\{t_i\}_{i=1}^{\infty}$ in T, there exists a sequence $\{B_i\}_{i=1}^{\infty} \in \mathfrak{A}$ such that $t_i \in B_i$ for each i.

Proof. There is a $1-1$ correspondence of T with $\{-1, 1\}^{\mathfrak{n}}$. By Lemma 3.2 the collection $\mathfrak{A}'$ of sequences of disjoint zero sets in $\{-1, 1\}^{\mathfrak{n}}$ fulfills the requirements of the theorem for $\{-1, 1\}^{\mathfrak{n}}$. The $1-1$ correspondence carries the collection $\mathfrak{A}'$ into the corresponding collection $\mathfrak{A}$ associated with T. Obviously $\mathfrak{A}$ has the desired properties.

4. *A Theorem of Hulanicki and a Generalization*

4.1. Definition. Let $\mathscr{A}$, $\mathscr{B}$ be collections of subsets of a space X. Then $\mathscr{A}$ is a weak base for $\mathscr{B}$ if and only if given a non-empty $B \in \mathscr{B}$ there is a non-empty $A \in \mathscr{A}$ such that $A \subset B$.

In what follows a measurable space $(X, \mathscr{B})$ is a space X and a σ-field $\mathscr{B}$. If $\{(X_t, \mathscr{B}_t) : t \in T\}$ is a family of measurable spaces then the product measurable space $(X, \mathscr{B})$ is the space $X = \mathbf{P}_{t \in T} X_t$ and the σ-field $\mathscr{B}$ generated by the cylinder sets $\pi_t^{-1}(M)$ where $M \in \mathscr{B}_t$, $t \in T$ (here π_t is the projection onto X_t).

In 1959, Hulanicki [10] proved the following theorem: Let $X = \mathbf{P}_{t \in T} X_t$, where $\{(X_t, \mathscr{B}_t : t \in T\}$ is a family of separable measurable spaces and $|T| \leqslant 2^{\mathfrak{n}}$. Then the product measurable space $(X, \mathscr{B})$ has a weak base $\mathscr{A}$ for the σ-field $\mathscr{B}$ for which $|\mathscr{A}| \leqslant \mathfrak{n}^{\aleph_0}$. Here a measurable space $(X, \mathscr{B})$ is separable if and only if there exists a set $A \subset X$ such that $|A| \leqslant \mathfrak{c}$ and $A \cap B \neq \emptyset$ for every $B \in \mathscr{B}$.

(Hulanicki was able to show that this theorem implies that there is an extension of Haar measure on the circle of character $2^{\mathfrak{c}}$.) In the proof of this theorem Hulanicki employed some sophisticated results from the theory of analytic sets (operations of Suslin) to show that the set he obtains is really a weak base for the σ-field $\mathscr{B}$.

In this section we will show that it is possible to obtain a generalization of Hulanicki's theorem by using only elementary facts from the theory of product and function spaces. We make the following observation for the case in which all $(X_t, \mathscr{B}_t)$, $t \in T$, are identical. If $\mathscr{B}$ is the σ-field generated by the sets $\pi_t^{-1}(M)$, $t \in T$, (where for each t, π_t is the projection mapping of $\mathbf{P}_{t \in T} X_t$ onto $X_t = X$, and M_t is a member of the given σ-field in X), then each member of $\mathscr{B}$ obviously depends on at most countably many coordinates. Thus if $\mathscr{A}$ is a collection of functions having the property that given a sequence of points $\{t_i\}_{i=1}^{\infty}$ in T and another sequence $\{x_i\}_{i=1}^{\infty}$ in X there is a function $f \in \mathscr{A}$ satisfying $f(t_i)=x_i$ for each i, it follows that $\mathscr{A}$ is a weak base for $\mathscr{B}$. We will now state and prove the generalized Hulanicki type theorem.

4.2. Theorem. Let $X=\mathbf{P}_{t\in T}X_t$, where $\{(X_t, \mathscr{B}_t) : t \in T\}$ is a family of measurable spaces, each having a weak base of cardinal at most $\mathfrak{n}^{\aleph_0}$, and $|T| \leqslant 2^{\mathfrak{n}}$. Then the product measurable space $(X, \mathscr{B})$ has a weak base $\mathscr{A}$ for the σ-field $\mathscr{B}$ for which $|\mathscr{A}| \leqslant \mathfrak{n}^{\aleph_0}$.

Proof: Consider the product measurable space $(Y^T, \mathscr{B})$ where each coordinate measurable space $(Y, \mathscr{C})$ satisfies $|Y| \leqslant \mathfrak{n}^{\aleph_0}$, and $\mathscr{C}$ is the σ-field of all subsets of Y. Let $\mathfrak{A}$ be the family of sequences of T satisfying conditions (i), (ii) of Theorem 3.3. Let $\mathscr{A} \subset Y^T$ be chosen as follows: For each element $\{B_i\}_{i=1}^{\infty}$ of $\mathfrak{A}$, choose one function from each equivalence class of functions $f : T \to Y$ constant on each B_j and determined by the relation $f \sim g$ if and only if $f=g$ on each B_j. It is clear that at most $\mathfrak{n}^{\aleph_0}$ functions are chosen. Repeating this procedure for each element of $\mathfrak{A}$ we see that the collection $\mathscr{A}$ of functions chosen in this manner is a weak base for the σ-field of the product measurable space $(Y^T, \mathscr{B})$ satisfying $|\mathscr{A}| \leqslant \mathfrak{n}^{\aleph_0}$.

For the general case, we observe that any $1-1$ correspondence of Y with a set of points corresponding to a weak base of a measurable space $(X_t, \mathscr{B}_t)$ is a measurable mapping of Y into X_t (see Hulanicki [10], Lemmas 3, 4, and 5). Thus for each t, let g_t be a $1-1$ (measurable) mapping of Y onto a weak base of X_t. Let $f : Y^T \to \mathbf{P}_{t\in T}X_t$ be the function defined by $f=\{g_t\}_{t\in T}$. Then f is measurable, is onto a weak base for the product measurable space of $(X, \mathscr{B})$ and $f(Y^T)$ has cardinal at most $\mathfrak{n}^{\aleph_0}$.

5. *Applications to Compact Abelian Groups*

As we have seen in Lemma 3.1, if $w(H) \leqslant \mathfrak{n}$ then H has at most $\mathfrak{n}^{\aleph_0}$ closed G_δ sets. Thus trivially there is a weak base for the Baire sets of H having cardinal at most $\mathfrak{n}^{\aleph_0}$ (Since there are at most $(\mathfrak{n}^{\aleph_0})^{\aleph_0}=\mathfrak{n}^{\aleph_0}$ Baire sets).

5.1. Lemma. Let $Y=\mathbf{P}_{t\in T}X_t$, where for each $t \in T$, X_t is a compact Hausdorff space. For each $t \in T$ let $\mathscr{B}_t$ be the σ-field of Baire sets in X_t. Let $\mathscr{B}$ be the product σ-field generated by the cylinder sets $\pi_t{}^{-1}(M_t)$ $(M_t \in \mathscr{B}_t)$. Then $\mathscr{B}$ is the σ-field of Baire sets in Y.

Proof: It is an elementary fact that points can be separated by the open subbasic cylinder sets $\pi_t{}^{-1}(U_t)$ since the coordinate spaces X_t are Hausdorff. Thus given two points $x, y \in Y$ there is a subbasic open set U containing x and not y. Since Y is compact and hence normal there is a continuous function f such that $f(x)=0$ and $f \equiv 1$ on the complement of U. Since U depended on only one coordinate f may be chosen to depend on only one coordinate. Thus by the Stone–Weierstrass theorem the algebra generated by the functions dependent on only one coordinate is uniformly dense in the continuous real valued functions. Since the Baire sets consist of the σ-field generated by the zero sets of the real valued continuous functions on Y it is immediate that they are also generated by the zero sets of the continuous functions that depend on only one coordinate (i.e. G_δ sets that are cylinder sets). This proves the lemma.

5.2. Corollary. Let $Y = \mathsf{P}_{t \in T} X_t$ where $|T| \leqslant 2^{\mathfrak{n}}$ and where for each $t \in T$, X_t is a compact Hausdorff space. For each $t \in T$, let $\mathscr{B}_t$ be the σ-field of Baire sets in X_t, and let $\mathscr{B}_t$ have a weak base of cardinal at most $\mathfrak{n}^{\aleph_0}$. Then the σ-field of Baire sets in Y has a weak base of cardinal at most $\mathfrak{n}^{\aleph_0}$.

Proof. Theorem 4.2 and Lemma 5.1

5.3. Corollary. Let $G = \mathsf{P}_{t \in T} H_t$, where each $H_t = H$, H is a compact Abelian group, $w(H) \leqslant \mathfrak{n}$, and $|T| \leqslant 2^{\mathfrak{n}}$. Then $w(G) \leqslant 2^{\mathfrak{n}}$ and there is a weak base for the Baire sets of G having cardinal at most $\mathfrak{n}^{\aleph_0}$.

5.4. Corollary. Let $G = \mathsf{P}_{t \in T} H_t$, where each $H_t = H$, H is a compact Abelian group, $w(H) \leqslant \mathfrak{n}$, and $|T| \leqslant 2^{2^{\mathfrak{n}}}$. Then there is a weak base for the Baire sets of G having cardinal at most $2^{\mathfrak{n}}$. (Here $w(G) \leqslant 2^{2^{\mathfrak{n}}}$.)

Proof. Observe that if $w(H) \leqslant \mathfrak{n}$, then $|H| \leqslant 2^{\mathfrak{n}}$. Thus in Corollary 5.2 replace $\mathfrak{n}$ by $\mathfrak{m} = 2^{\mathfrak{n}}$.

5.5. Corollary. Let $G = \mathsf{P}_{t \in T} H_t$, where each $H_t = H$, H is a compact Abelian group and $w(H) \leqslant \mathfrak{n}$.

(a) If $|T| \leqslant 2^{\mathfrak{n}}$ then G contains a dense pseudocompact subgroup J satisfying $|J| \leqslant \mathfrak{n}^{\aleph_0}$; necessarily J has Haar outer measure one.

(b) If $|T| \leqslant 2^{2^{\mathfrak{n}}}$ then G contains a dense pseudocompact subgroup J satisfying $|J| \leqslant 2^{\mathfrak{n}}$; necessarily J has Haar outer measure one.

Proof. A theorem of Comfort and Ross [1] states that a totally bounded topological group G is pseudocompact if and only if each non-empty Baire subset of $\bar{G}$ meets G, where $\bar{G}$ is the Weil completion of G. Select one element from each member of a weak base for the Baire sets of G. It is clear that the group J generated by this set is pseudocompact. Part (a) is immediate from Corollary 5.1 and part (b) is immediate from Corollary 5.2. To complete the proof we note that a set $A \subset G$ has Haar outer measure one if and only if $A \cap B \neq \emptyset$ for each Baire set B of positive measure.

This corollary generalizes to arbitrary compact Abelian topological groups because of the following theorem of Vilenkin [17]: Let G be a compact Abelian group. For some cardinal number $\mathfrak{m}$, there is a continuous mapping of $\{-1, 1\}^{\mathfrak{m}}$ onto G; $\mathfrak{m}$ can be taken to be max $[\aleph_0, r]$ where r is the rank of the character group of G. We note here that if $|G| > \aleph_0$ then $r = |G|$ (see [6] A.11–16).

5.6. Theorem. Let G be a compact Abelian topological group satisfying $w(G) = 2^{\mathfrak{n}}$. Then

(i) G has a weak base for its Baire sets with cardinal at most $\mathfrak{n}^{\aleph_0}$,

(ii) G contains a dense pseudocompact subgroup J such that $|J| \leqslant \mathfrak{n}^{\aleph_0}$; necessarily J has Haar outer measure one.

Proof. Clearly by Pontryagin's theorem $|G| = 2^{\mathfrak{n}}$ so by Vilenkin's theorem G is the continuous image of $G' = \{-1, 1\}^{\mathfrak{m}} = \mathsf{P}_{t \in T}\{-1, 1\}_t$, where $|T| = \mathfrak{m} = 2^{\mathfrak{n}}$. Thus Corollary 5.3 applies to G', so G' contains a weak base for the Baire sets having cardinal at most $\mathfrak{n}^{\aleph_0}$. However under a continuous mapping weak bases for Baire sets go over onto weak bases for Baire sets (i.e., each closed G_δ in G is a continuous image of a closed G_δ in G', and the G_δ sets generate the σ-field of Baire sets). Thus there is a weak base $\mathscr{B}$ for the Baire sets in G having cardinal at most $\mathfrak{n}^{\aleph_0}$. This proves (i). For (ii), let J be the group generated by the set obtained by selecting one point from each element of the weak base $\mathscr{B}$.

Remark. This theorem complements the theorem of Hartman and Hulanicki that we have quoted in section 2. We emphasize that our Theorem 5.6 is true even without the continuum hypothesis. We make now the following observations:

(1) If $\mathfrak{n} = \mathfrak{n}^{\aleph_0}$, then our theorem really strengthens the theorem of Hartman and Hulanicki.

(2) If $\mathfrak{n} = \aleph_0$ our theorem appears to be weaker than the theorem of Hartman and Hulanicki. However we note the following: there does not exist a pseudocompact Abelian group of cardinal $\aleph_0$. This is due to the fact that a pseudocompact group necessarily has Haar outer measure one in its Weil completion (which is compact). However a countable subset of an infinite compact group can only have Haar measure zero.

Note. We mention that Theorem 5.4 contains Theorem 2 in Hulanicki [10].

6. *Statement of the Main Problem*

Kakutani and Oxtoby [13] proved that Haar measure in a compact metric group may be extended to a much larger σ-field of subsets of the group and still remain invariant under group translation and inversion. To be more precise we introduce the following definition.

6.1. Definition. Let X be any set, $\mathscr{S}$ any σ-algebra of subsets of X, and μ any measure defined on $\mathscr{S}$. Then the character of the measure space $(X, \mathscr{S}, \mu)$ is the smallest cardinal number $\mathfrak{m}$ for which there exists a subfamily $\mathscr{T}$ of $\mathscr{S}$ such that $|\mathscr{T}| = \mathfrak{m}$ and such that for every $S \in \mathscr{S}$ and every $\varepsilon > 0$, there exists a set $T \in \mathscr{T}$ satisfying $\mu(S \triangle T) < \varepsilon$. Any such subfamily will be called a basis for $(X, \mathscr{S}, \mu)$.

It is well known that the character of the Haar measure space of a compact infinite metric group is $\aleph_0$ (see 6.3). Kakutani and Oxtoby showed that there is an extension of Haar measure with character $2^{\mathfrak{c}}$.

This is a surprising result simply because $2^{\mathfrak{c}}$ is the cardinal of the power set of the group. HEWITT and ROSS [6] 16.3, noticed that the method of proof used actually worked for all compact metric groups.

An obvious generalization of this theorem would be the following:

6.2. Conjecture. Let G be a compact topological group satisfying $w(G)=\mathfrak{n}$. Then there is a translation and inversion invariant extension of Haar measure having character $2^{2^{\mathfrak{n}}}$.

This conjecture should be compared with the following observation.

6.3. Theorem. Let G be an infinite compact topological group. Then the character of $(G, \mathscr{M}, m)$ is at most equal to $w(G)$. Here m is Haar measure, and $\mathscr{M}$ consists of the Haar measurable sets.

Proof. Consider the collection $\mathscr{O}$ of finite unions of open sets contained in a basis $\mathscr{B}$ of the topological space G where $|\mathscr{B}|=w(G)$. Since $w(G)$ is infinite it is clear that $|\mathscr{O}|=w(G)$. Let T be a measurable set and suppose that $m(T)>0$. By regularity of Haar measure there is a compact set $F\subset T$ and an open set $O\supset T$ such that $m(O \triangle T)<\varepsilon$ for some preassigned $\varepsilon>0$. Since F is compact there is a finite number of sets in $\mathscr{B}$ that cover F and are contained in O. Let U denote their union. Obviously $U\in\mathscr{O}$ and $m(U \triangle T)<\varepsilon$, thus the character is at most $|\mathscr{O}|=w(G)$.

If G is Abelian, then the character $\mathfrak{m}$ of $(G, \mathscr{M}, m)$ is equal to $w(G)$ since

$$\mathfrak{m}=\dim L_2(G, \mathscr{M}, m)=|G|=w(G).$$

The first equality follows from the following fact: Let $(X, \mathscr{M}, \mu)$ be a measure space such that $\mu(X)=1$. Let $\mathfrak{d}$ be the dimension of $L_2(X, \mathscr{M}, \mu)$ and let $\mathfrak{m}$ be the character of $(X, \mathscr{M}, \mu)$. If $\mathfrak{d}$ is finite, then $\mathfrak{m}=2^{\mathfrak{d}}$. If $\mathfrak{d}$ is infinite then $\mathfrak{m}=\mathfrak{d}$. (See [6], 16.12.)

In the present paper we shall prove the following theorem.

6.4. Theorem. Let G be a compact connected Abelian group satisfying $w(H)=\mathfrak{n}$. Then there exists a translation and inversion invariant extension of Haar measure on H of character $2^{2^{\mathfrak{n}}}$.

Note that if $w(H)=\mathfrak{n}$ then $|H|=2^{\mathfrak{n}}$, so that $2^{2^{\mathfrak{n}}}$ is the cardinal of the power set of H.

Remark. The method of proof that we will employ is essentially due to KAKUTANI and KODAIRA [12]. In their paper they showed that there is a translation and inversion invariant extension of Haar measure on the circle of character $\mathfrak{c}$. HULANICKI, using Theorem 1 of [10] (which we have quoted in Section 4), showed that the same method may be used to get an extension on the circle of character $2^{\mathfrak{c}}$. It turns out that our Corollary 5.4 a related existence theorem, is not quite enough to prove Theorem 6.4. In the next section we will exhibit a weak base, for the closed G_δ sets of positive measure of G (of Corollary 5.4), that will have the properties necessary for the proof of our theorem.

7. *A Weak Base for Baire Sets of Positive Measure*

Let H be a compact Abelian group satisfying $w(H) \leqslant \mathfrak{n}$. Let $G = \mathbf{P}_{t \in T} H_t$ where each $H_t = H$ and $|T| = 2^{2^{\mathfrak{n}}}$. Let $\mathcal{N}_t = \mathcal{N}$ be the collection of closed G_δ sets of $H_t = H$. It is clear from Lemma 3.1 that $|\mathcal{N}| \leqslant \mathfrak{n}^{\aleph_0}$.

7.1. Definition. *A $(T, \mathcal{N})$-cylinder $M \subset G$ is a set of the form*

$$M = \bigcap\nolimits_{t \in B} \pi_t^{-1}(N_t)$$

where $B \subset T$, and each $N_t = N$ for some fixed $N \in \mathcal{N}$. For convenience we will use the notation (B, N) to mean $\bigcap_{t \in B} \pi_t^{-1}(N_t)$. If B consists of the single point t we will write (t, N) to represent $\pi_t^{-1}(N)$.

Intuitively (B, N) is a cylinder of cross section N in each of B coordinates and of cross section H in the remaining ones.

Let T be as above. As we noted previously, there is a $1-1$ correspondence of T with $\{-1, 1\}^{2^{\mathfrak{n}}}$. Let $\mathcal{G}$ be the collection of subsets of T corresponding to the closed G_δ sets in $\{-1, 1\}^{2^{\mathfrak{n}}}$. Let $\mathfrak{A}$ be the collection of sequences in $\mathcal{G}$ of pairwise disjoint sets in T satisfying properties (i) and (ii) of Theorem 3.3.

7.2. Definition. An $(\mathfrak{A}, \mathcal{N})$-cylinder $M \subset G$ is a set of the form $M = \bigcap_{i=1}^{\infty} (B_i, N_i)$, where each $N_i \in \mathcal{N}$ and $\{B_i\}_{i=1}^{\infty} \in \mathfrak{A}$.

Let $\mathcal{C}_\delta$ be the collection of all $(\mathfrak{A}, \mathcal{N})$-cylinders. It is clear that $|\mathcal{C}_\delta| \leqslant |\mathcal{G} \times \mathcal{N}|^{\aleph_0} \leqslant 2^{\mathfrak{n}}$.

In what follows we employ the standard definitions of measurable spaces, measure spaces and product measure spaces. (See for example Halmos [4], Hewitt and Ross [6]).

7.3. Lemma. *Let $G = H_1 \times H_2$, where H_1 and H_2 are compact. Let $A \times B \subset H_1 \times H_2$ be closed (compact) and be such that A consists of a single point x_1. Suppose $A \times B \subset U$ where U is open. Then there is an open rectangle $U_1 \times U_2$ satisfying $A \times B \subset U_1 \times U_2 \subset U$. (A rectangle is a set of the form $A_1 \times A_2$ where $A_1 \subset H_1$ and $A_2 \subset H_2$.)*

Proof. Easy.

Consider the product measure space of the measure spaces $\{(X_t, \mathcal{M}_t, \mu_t):$ $t \in T\}$. Let $M \subset \mathbf{P}_{t \in T} X_t$. For $\beta \in T$, denote the complement of $\{\beta\}$ in T by $\bar{\beta}$. Let $x_{\bar{\beta}} \in \mathbf{P}_{t \neq \beta} X_t$. We define the $x_{\bar{\beta}}$ section in M as the set of points

$$M_{x(\bar{\beta})} = \{x_\beta | (x_\beta, x_{\bar{\beta}}) \in M\}.$$

The collection of sets of this form is called the collection of $X_{\bar{\beta}}$ sections. It is an elementary fact of measure theory that if $(X \times Y, \mathcal{S} \times \mathcal{T}, \mu \times v)$ is a measure space where μ, v are σ-finite and if $E \subset X \times Y$ is measurable, then E has measure zero if and only if almost every X-section has measure zero. It is an immediate consequence of this fact that if $M \subset \mathbf{P}_{t \in T} X_t$ has positive measure, then for every β there is an $X_{\bar{\beta}}$-section $M_{x_{\bar{\beta}}}$ which has positive measure in X_β.

From now on for each $t \in T$, $(X_t, \mathscr{M}_t, \mu_t)$ is the Haar measure space of the compact group $X_t = H_t = H$ Satisfying $w(H) = \mathfrak{n}$. G is the topological product group $G = \mathbf{P}_{t \in T} H_t$ where $|T| = 2^{2^{\mathfrak{n}}}$.

If M is a closed G_δ in G then it is immediate from compactness that M depends on only countably many coordinates. (Compare with Lemma 5.1.) In fact we can regard each of the open sets that appear in the intersection as depending on only finitely many coordinates. We note then that each closed G_δ set M may be written in the form

$$M = M_{t_1 t_2} \ldots \times \mathbf{P}_{t \notin \{t_i\}} H_t.$$

7.4. Lemma. Let M be a closed G_δ in G having positive Haar measure. Then for each fixed coordinate $\beta \in \{t_i\}_{i=1}^{\infty}$, where the t_i are the coordinates that M depends on, there is a subset $N \subset M$ that is a closed G_δ of the form $N = \bigcap_{n=1}^{\infty} (t_n, N_{t_n})$ where each N_{t_n} is a closed G_δ in H_{t_n} and N_β has positive Haar measure in H_β.

Proof. For simplicity we will assume $t_i = i$ $(i = 1, 2, \ldots)$ and that $\beta = 1$. Since M has positive Haar measure there is an $H_{\bar{1}}$-section $M_{x_{\bar{1}}}$ having positive Haar measure in H_1. Regularity of Haar measure implies that there is a closed G_δ set $N_1 \subset M_{x_{\bar{1}}}$ having positive Haar measure in H_1. We will consider the rectangular set $N_1 \times \{x_{\bar{1}}\} \subset M \subset G$. Since M is a closed G_δ, $M = \bigcap_{r=1}^{\infty} U_r$ and obviously $N_1 \times \{x_{\bar{1}}\} \subset U_r$ for each r. By Lemma 7.3 there are rectangular open sets $V_r \times V_{\bar{r}}$ satisfying $N_1 \times \{x_{\bar{1}}\} \subset V_r \times V_{\bar{r}} \subset U_r$ for each r. Replace $V_{\bar{r}}$ by a basic open set W_r satisfying $\{x_{\bar{1}}\} \in W_r = \bigcap_{j=2}^{n_r} (j, W_{j,r}) \subset V_{\bar{r}}$ where $W_{j,r}$ is a neighbourhood in H_j of x_j. Since each H_i is normal we see that for each r there is a zero set Z_r satisfying $N_1 \subset Z_r \subset V_r$. Furthermore for $j = 2, \ldots, n_r$ there are zero sets $Z_{j,r}$ satisfying

$$x_j \in Z_{j,r} \subset W_{j,r}.$$

Thus for each r, there is a relationship of the form

$$N \times \{x_{\bar{1}}\} \subset (1, Z_r) \cap [\bigcap_{j=2}^{n_r} (j, Z_{j,r})] \subset U_r.$$

Thus there is a zero set

$$\bigcap_{r=1}^{\infty} \{(1, Z_r) \cap [\bigcap_{j=1}^{n_r} (j, Z_{j,r})]\} \subset \bigcap_{r=1}^{\infty} U_r = M$$

which fulfills the requirements of the theorem.

7.5. Corollary. Let $\beta \in T$ be fixed. Let $\mathscr{F}_\beta$ denote the collection of closed G_δ sets in G of the form $N = \bigcap_{i=1}^{\infty} (t_i, N_{t_i})$, where $N_{t_i} \in \mathscr{N}$ for each i, $\beta \in \{t_i\}_{i=1}^{\infty}$, and N_β has positive Haar measure in H_β. Then $\mathscr{F}_\beta$ is a weak base for the closed G_δ sets in G having positive measure.

Proof. Let M be a closed G_δ set in G having positive measure. Then M has the form $M = M_{t_1 t_2} \ldots \times \mathbf{P}_{t \notin \{t_i\}_{i=1}^{\infty}} H_t$. Without loss of generality we may assume that $\beta \in \{t_i\}_{i=1}^{\infty}$. Thus apply Lemma 7.4.

7.6. Theorem. Let $\beta \in T$ be fixed. Let $\mathscr{P}_\beta \subset \mathscr{C}_\delta$ consist of those $(\mathfrak{A}, \mathscr{N})$-cylinders of the form $\bigcap_{n=1}^{\infty} (B_n, N_n)$ that satisfy $\beta \in B_i$ for some i and N_i has positive Haar measure in H for this i. Then $\mathscr{P}_\beta$ is a weak base for the closed G_δ sets in G having positive Haar measure.

Proof. Let M be a closed G_δ in G with positive Haar measure. By Corollary 7.5 there is a member of $\mathscr{F}_\beta$ contained in M. Suppose it is $\bigcap_{i=1}^{\infty} (t_i, N_i)$. Since $\mathfrak{A}$ was chosen so that there is a sequence $\{B_i\}_{i=1}^{\infty} \in \mathfrak{A}$ of pairwise disjoint subsets of T satisfying $t_i \in B_i$ for each i, it is clear that $\bigcap_{i=1}^{\infty} (B_i, N_i) \subset \bigcap_{i=1}^{\infty} (t_i, N_i) \subset M$ and obviously $\bigcap_{i=1}^{\infty} (B_i, N_i) \in \mathscr{P}_\beta$. This proves the theorem.

It is clear from the construction of $\mathscr{P}_\beta$ that $|\mathscr{P}_\beta| \leqslant 2^{\mathfrak{n}}$ and if $P \in \mathscr{P}_\beta$ then $\pi_\beta(P)$ has positive Haar measure in H_β and is a closed G_δ there (π is the projection onto H_β).

Remark. It is not difficult to show that $\mathscr{C}_\delta$ is a weak base for the closed G_δ sets of G. We have omitted the proof of this because we do not actually use this fact to prove Theorem 6.4. The proof is very similar to the proof of Theorem 7.6.

Remark. Theorem 7.6 is the crucial step in proving that we can actually get the desired extension of Haar measure on a compact connected Abelian group. The remainder of the proof is essentially a repetition of the paper of Kakutani and Kodaira [12] (of course in a more general setting). It will become clear shortly that in some sense every compact connected group looks like the circle group. In fact, as we shall see, the important properties that Kakutani and Kodaira utilized in their proof were the connectedness of the circle and the existence of sufficiently many independent elements of infinite order. Thus with the present knowledge of the structure of infinite compact Abelian groups (due to Hulanicki [8], [9], and others) it will not be difficult to imitate the proof of Kakutani and Kodaira.

Finally we remark that if we wished only to get an extension of Haar measure of character $2^{\mathfrak{n}}$ (where $w(H) = \mathfrak{n}$) by using the method of Kakutani and Kodaira, Theorem 7.6 would not be necessary. We note the following: Let H be as before and let $G = \mathbf{P}_{t \in T} H_t$ where $|T| = 2^{\mathfrak{n}}$. Then $\widehat{G} = \mathbf{P}^*{}_{t \in T} \widehat{H}_t$ and $|\widehat{G}| = 2^{\mathfrak{n}}$. Thus as in the proof of Lemma 3.1 we may conclude that there are at most $2^{\mathfrak{n}}$ closed G_δ sets in G with positive measure. Furthermore, if M is a closed G_δ set in G having positive Haar measure then for each $t \in T$, $\pi_t(M)$ contains a closed G_δ set in H_t having positive Haar measure. With this information one could imitate the remaining sections of this paper and produce a proof of the fact that there is a translation and inversion invariant extension of Haar measure of character $2^{\mathfrak{n}}$ for the Haar measure space of H where $w(H) = \mathfrak{n}$.

8. *Existence of Large Sets of Independent Elements*

In the remainder of this paper Abelian groups will be written with additive notation.

A well known theorem of Gelfand and Silov states that for a locally compact Abelian group G if $M \subset G$ has positive Haar measure then $M-M$ contains the identity in its interior. This implies the following lemma.

8.1. Lemma. Let G be a compact Abelian group. Let $M \subset G$ be a Haar measurable set having positive Haar measure. Then the group $[M]$ generated by M is open and hence has finite index in G.

8.2. Definition. A finite subset $x_1, \ldots, x_k$ of an Abelian group G is said to be independent if it does not contain 0 and if

$$n_1x_1 + \ldots + n_kx_k = 0 \quad (n_i \text{ all integers})$$

implies that

$$n_1x_1 = n_2x_2 = \ldots = n_kx_k = 0.$$

An infinite subset I is independent if every finite subset of I is independent.

We note that if the $\{x_i\}_{i=1,\ldots,k}$ are of infinite order then this reduces to the standard definition of independence, i.e., $n_1 = n_2 = \ldots = n_k = 0$.

8.3. Lemma. Let G be an Abelian group and let $M \subset G$. Let $L \subseteq M$ be a set of elements of infinite order in G. If every element of infinite order in M is dependent on L and if $[M]$ has finite index in G, then every element of infinite order in G is dependent on L.

Proof. Easy.

The following structure theorem, due to Hulanicki [8], [9] and others (see also [6], 25.23), plays an important role in our proof: Let G be a connected compact Abelian group satisfying $w(G) = \mathfrak{m}$. Then G is algebraically isomorphic with

$$Q^{2^{\mathfrak{m}}*} \times \mathsf{P}^*_{p \in P}\, Z(p^\infty)^{\mathfrak{b}_p*}$$

($*$ denotes weak direct product) where P is the set of all primes, and each cardinal number $\mathfrak{b}_p$ is finite or is $2^{\mathfrak{e}_p}$ for an infinite cardinal $\mathfrak{e}_p \leqslant \mathfrak{m}$.

8.4. Definition. Let G be an Abelian group and let L be a maximal independent set of elements of infinite order in a set $M \subset G$. We shall then say that L is a basis for the elements of infinite order in M. If $M = G$ the cardinal number of L is called the torsion free rank $r_0(G)$ of G.

With this convention in mind we state the following facts (see [6], A11–14).

(a) All bases of elements of infinite order in G have the same number of elements.

(b) If G is torsion free, if $|G|>\aleph_0$, and if L is a basis for the elements of infinite order in G, then $|G|=|L|$.

(c) Let F be the torsion subgroup of G. Then $r_0(G)=r_0(G/F)=r(G/F)$ where r_0 denotes torsion free rank and r denotes rank.

8.5. Lemma. Let G be an Abelian group. Let $M\subset G$ and let $[M]$ have finite index in G. Then M contains a basis for the elements of infinite order in G.

Proof. We may suppose $r_0(G)\neq 0$. Clearly the collection of independent sets of elements of infinite order in M has finite character (à la Kelley p. 32), so that Tukey's Lemma applies here. Thus there is a basis L for the elements of infinite order in M. However $[M]$ has finite index in G, so Lemma 8.3 applies, and we conclude that L is a basis for the elements of infinite order in G.

8.6. Theorem. Let G be a compact Abelian topological group. Let $M\subset G$ be a set of positive Haar measure. Then M contains a basis for the elements of infinite order in G.

Proof. By Lemma 8.1 $[M]$ has finite index in G, so by Lemma 8.5 M contains a basis for the elements of infinite order in G.

8.7. Corollary. Let G be a compact connected Abelian topological group satisfying $w(G)=\mathfrak{n}$. Then every closed G_δ set $M\subset G$ having positive Haar measure contains a basis L for the elements of infinite order in G and $|L|=2^{\mathfrak{n}}$.

Proof. By Theorem 8.6 each closed G_δ set $M\subset G$ having positive measure contains a basis L for the elements of infinite order in G. We note that by the structure theorem of Hulanicki one such basis has $2^{\mathfrak{n}}$ elements. Thus $|L|=2^{\mathfrak{n}}$.

8.8. Lemma. Let G be a compact connected Abelian group satisfying $w(G)=\mathfrak{n}$. Let $\{M_\alpha : \alpha<\omega_{\mathfrak{m}},\ \mathfrak{m}=2^{\mathfrak{n}}\}$ be a well ordered sequence of closed G_δ sets of positive Haar measure in G. Then there exists a well ordered set $\{x_\alpha : \alpha<\omega_{\mathfrak{m}}\}$ of independent elements of infinite order such that $x_\alpha\in M_\alpha$ for each $\alpha<\omega_{\mathfrak{m}}$ (the M_α's are not necessarily distinct).

Proof. By Corollary 8.7 for each α there is a basis L_α for the elements of infinite order in G, where $L_\alpha\subset M_\alpha$ and $|L_\alpha|=2^{\mathfrak{n}}$.

Let $x_1\in L_1$ be arbitrary. Let $\gamma<\omega_{\mathfrak{m}}$ and suppose that we have chosen a sequence $\{x_\alpha : \alpha<\gamma\}$ of independent elements of infinite order satisfying $x_\alpha\in L_\alpha$ for each $\alpha<\gamma$.

It is clear that $\{x_\alpha : \alpha<\gamma\}$ is not a basis for the elements of infinite order in G since each such basis contains $2^{\mathfrak{n}}$ elements and $|\gamma|<2^{\mathfrak{n}}$. Thus there is $y\in G$ independent of $\{x_\alpha : \alpha<\gamma\}$ and y is of infinite order. Since L_γ is a basis for the elements of infinite order y is dependent on L_γ. Thus there is a relation of the form $my = n_1 z_1 + \ldots + n_k z_k$ where $m \neq 0$ and where for each i, $z_i \in L_\gamma$ and n_i is an integer. Clearly if each z_i is dependent on $\{x_\alpha : \alpha < \gamma\}$ then so is y. Thus at least one z_i is independent of $\{x_\alpha : \alpha < \gamma\}$. Choose one such z_i and call it x_γ. Obviously $x_\gamma \in L_\gamma \subset M_\gamma$ and the transfinite induction is complete.

Remark. Lemma 8.8 is true in a more general situation. The same induction will work because of Theorem 8.6 if the M_α are measurable with positive measure, m is at most equal to the cardinal of a maximal independent set of elements of infinite order, and G is compact Abelian (with no other restrictions).

9. *An Embedding Lemma*

9.1. Definition. An Abelian group G is said to be divisible if and only if given $x \in G$ and n a nonzero integer, there is $y \in G$ such that $x = ny$ (in additive notation).

The next lemma in a slightly less general setting was used without proof in Kakutani and Kodaira [12]. It is crucial to our proof of the extension of Haar measure. It will enable us to embed a compact connected Abelian group into a large product group in such a manner that the embedded group has Haar outer measure one in the large group. We note that this is an algebraic embedding and not a topological embedding. It is well known (see [16], p. 44) that every homomorphism ϕ of a group $H \subset G$ into a divisible group D may be extended to a homomorphism ϕ' of G into D. We use this fact in the proof of the Lemma.

9.2. Lemma. Let $G = H \times J$ where H is an Abelian group. Let $K \subset H$ be a proper subgroup. Suppose that φ is an isomorphism of K into G satisfying $\pi_H \varphi(x) = x$ for all $x \in K$, where π_H is the projection of G onto H. Then φ can be extended to an isomorphism φ' of H into G satisfying $\pi_H \varphi'(x) = x$ for all $x \in H$.

Proof. We observe that the isomorphism φ of $K \subset H$ into $G = H \times J$ may be written in the form $\varphi(x) = (x, \theta(x))$ where θ is a homomorphism of K into J. By the remark in the above paragraph θ may be extended to a homomorphism θ' of H into J. Let $\varphi'(x) = (x, \theta'(x))$ for all $x \in H$. Clearly φ' is the desired isomorphism.

10. *Induced Measure Spaces*

The concept of an induced measure space was introduced by Doob [3]. There is also a discussion of this type of measure space in Kakutani and Kodaira [12]. Since we use an induced measure space in the proof of Theorem 6.4, for completeness we will introduce the elementary facts about these spaces that we shall need.

Let $(S, \mathscr{M}, m)$ be a measure space for which $m(S) = 1$. It is elementary that a subset A has outer measure one in S if and only if $A \cap M \neq \emptyset$ for every $M \in \mathscr{M}$ satisfying $m(M) > 0$. Thus let $A \subset S$ have outer measure one. Let

$$\mathscr{M}^{\sim} = \{M^{\sim} : M^{\sim} = A \cap M,\ M \in \mathscr{M}\}.$$

Then $\mathscr{M}^{\sim}$ is a σ-field of subsets of A.

10.1. Definition. The induced measure $m^{\sim}$ on $\mathscr{M}^{\sim}$ is given by

$$m^{\sim}(M^{\sim}) = m(M)$$

if $M^{\sim} = A \cap M$, $M \in \mathscr{M}$.

It is easy to verify that $m^{\sim}$ is a well defined measure on the σ-field $\mathscr{M}^{\sim}$ (i.e., if $M^{\sim} = M_1 \cap A = M_2 \cap A$ then $m(M_1 \triangle M_2) = 0$).

10.2. Definition. $(A, \mathscr{M}^{\sim}, m^{\sim})$ is called the measure space induced on A by $(S, \mathscr{M}, m)$.

The following lemma was stated in Kakutani and Kodaira [12] and it is a crucial tool for the proof of our theorem.

10.3. Lemma. The measure space $(A, \mathscr{M}^{\sim}, m^{\sim})$ has the same character as the measure space $(S, \mathscr{M}, m)$.

Proof. Easy with the following observation:

Let T, $M \in \mathscr{M}$ and let $T^{\sim}$, $M^{\sim}$ be the corresponding sets in $\mathscr{M}^{\sim}$. It is an elementary fact that the identities

$$T^{\sim} \triangle M^{\sim} = (T \triangle M) \cap A = (T \triangle M)$$

hold.

11. *Embedding Into a Large Product Group*

11.1. Lemma. Let H be a compact connected Abelian group satisfying $w(H) = \mathfrak{n} \geqslant \aleph_0$. Let $G = \mathbf{P}_{t \in T} H_t$ where each $H_t = H$ and $|T| = 2^{2^{\mathfrak{n}}}$. Fix the coordinate $\beta \in T$. Then there is a set $V \subset G$ of independent elements of infinite order satisfying

(i) V has Haar outer measure one

(ii) $\pi_\beta|_V$ is one to one.

Proof. In Theorem 7.6 we have shown that $\mathscr{P}_\beta$ is a weak base for the G_δ sets of positive measure in G, $|\mathscr{P}_\beta| \leqslant 2^{\mathfrak{n}}$, and if $N \in \mathscr{P}_\beta$ then $\pi_\beta(N)$ has positive Haar measure in H_β and is a closed G_δ there.

Well-order the elements of $\mathscr{P}_\beta$ in a well ordering of type $\omega_{\mathfrak{m}}$, $\mathfrak{m} = 2^{\mathfrak{n}}$, i.e. $\{M_\alpha : \alpha < \omega_{\mathfrak{m}}\}$. By Lemma 8.8 (with G replaced by H) for each $\alpha < \omega_{\mathfrak{m}}$ we may select $x_\alpha \in \pi_\beta(M_\alpha)$ such that $\{x_\alpha : \alpha < \omega_{\mathfrak{m}}\}$ is an independent set of elements of infinite order. For each $\alpha < \omega_{\mathfrak{m}}$, let $y_\alpha \in \pi_\beta^{-1}(x_\alpha) \cap M_\alpha$ be selected arbitrarily. Let

$$V = \bigcup_{\alpha < \omega_{\mathfrak{m}}} \{y_\alpha\}.$$

Obviously V has Haar outer measure one in G since $V \cap M \neq \emptyset$ for each $M \in \mathscr{P}_\beta$ and hence by regularity of Haar measure $V \cap B \neq \emptyset$ for each Baire set B of positive measure. It is clear that $\pi_\beta|_V$ is 1–1 since the sets $\{x_\alpha : \alpha < \omega_\mathfrak{m}\}$ and $\{y_\alpha : \alpha < \omega_\mathfrak{m}\}$ consist of independent elements. This completes the proof of the lemma.

Let V_G be the group generated by V and let K be the group generated by $\pi_\beta(V)$. It is clear that π_β is an isomorphism of V_G onto K since these are free groups on the elements of V and $\pi_\beta(V)$ respectively and π_β is 1–1 on the generators of the free groups. Thus the inverse of π_β induces an isomorphism φ of $K \subset H$ into G, such that $\varphi(K)$ has Haar outer measure one in G.

It is well known (see [6], 24.25) that for a compact Abelian group G, the statement G is connected is equivalent to the statement G is divisible. Thus we have the following lemma.

11.2. Lemma. Let H be a compact connected Abelian group for which $w(H) = \mathfrak{n}$. Let $G = \mathbf{P}_{t \in T} H_t$ where each $H_t = H$ and $|T| = 2^{2^\mathfrak{n}}$. Fix the coordinate $\beta \in T$. Then there is an algebraic isomorphism φ' of H_β into G such that

(i) $\varphi'(H_\beta)$ has Haar outer measure one in G

(ii) $\pi_\beta \varphi'(x) = x$ for each $x \in H_\beta$.

Proof. By the above discussion there is an isomorphism φ of $K \subset H_\beta$ into G satisfying $\pi_\beta\varphi(x) = x$ for each $x \in K$. Clearly $J = P_{t \neq \beta} H_t$ is divisible. Thus the present lemma follows from Lemma 9.2.

12. *The Extension of Haar Measure*

For convenience we will restate here Theorem 6.4.

12.1 Theorem. Let G be a compact connected Abelian group satisfying $w(H) = \mathfrak{n}$. Then there exists a translation and inversion invariant extension of Haar measure on H of character $2^{2^\mathfrak{n}}$.

Proof. Embed H in $G = \mathbf{P}_{t \in T} H_t$ where each $H_t = H$ and $|T| = 2^{2^\mathfrak{n}}$, i.e., send H into H_β for some fixed $\beta \in T$. Let φ' be the isomorphism of H_β into G of Lemma 11.2, so that $m_G^*(\varphi'(H_\beta)) = 1$ where m_G is Haar measure on G and m_G^* is Haar outer measure on G.

We will now consider the measure space $(\varphi'(H_\beta), \mathscr{M}^\sim, m^\sim)$ induced by $(G, \mathscr{M}, m_G)$, where $\mathscr{M}$ is the σ-field of Haar measurable sets. By Lemma 10.3 the character of $(\varphi'(H_\beta), \mathscr{M}^\sim, m_G^\sim)$ is equal to the character of $(G, \mathscr{M}, m_G)$. We may further observe that the character of the latter measure space is $2^{2^\mathfrak{n}}$ simply because $\widehat{G} = \mathbf{P}^*_{t \in T} \widehat{H}_t$ has cardinality $2^{2^\mathfrak{n}}$. It is clear that $m_G^\sim$ is translation and inversion invariant on $\varphi'(H_\beta)$ since m_G is on G.

We observe that since φ' is an algebraic isomorphism there is associated with $\mathscr{M}^\sim$ the σ-field $\mathscr{H} = \{\varphi'^{-1}(M) : M \in \mathscr{M}^\sim\}$ of subsets of H_β. Thus we may define a measure h on H by

$$h(M)=m_G^{\sim}(\varphi'(M)) \text{ for } M \in \mathscr{H}.$$

Thus $(H_\beta, \mathscr{H}, h)$ is a measure space. It is clear that h is translation and inversion invariant. Obviously the character of $(H_\beta, \mathscr{H}, h)$ is $2^{2^{\mathfrak{n}}}$. Thus to complete the proof we must show that h, restricted to $\mathscr{M}_\beta$ where $\mathscr{M}_\beta$ is the σ-field of Haar measurable sets in H_β, is just Haar measure. Denote Haar measure in H_β by m_β.

Let $M_\beta \subset H_\beta$ be a Haar measurable set. Let

$$M=\pi_\beta^{-1}(M_\beta)=M_\beta \times \mathsf{P}_{t \neq \beta} H_t.$$

Then clearly M is a Haar measurable subset of G and

$$m_G(M)=m_\beta(M_\beta).$$

We may now observe that $M \cap \varphi'(H_\beta) \in \mathscr{M}^{\sim}$ and

$$m_G^{\sim}(M \cap \varphi'(H_\beta))=m_G(M)=m_\beta(M_\beta).$$

Furthermore, since φ' is an isomorphism of H_β into G satisfying $\pi_\beta\varphi'(x)=x$ for $x \in H_\beta$ it follows that $M \cap \varphi'(H_\beta)=\varphi'(M_\beta)$. Thus $M_\beta \in \mathscr{H}$ and furthermore

$$h(M_\beta)=m_G^{\sim}(\varphi'(M_\beta))=m_G^{\sim}(M \cap \varphi'(H_\beta))=m_G(M)=m_\beta(M_\beta).$$

Thus h is the desired extension of Haar measure on H_β.

The theorem is proved since it is obvious that we can now induce a measure space on H with the desired properties via the algebraic isomorphism of the embedding.

12.2. Corollary. Let H be a compact locally connected Abelian group satisfying $w(H)=\mathfrak{n}$. Then there is a translation and inversion invariant extension of Haar measure on H of character $2^{2^{\mathfrak{n}}}$.

Proof. The component C of the identity in H is a compact open subgroup satisfying $w(C)=\mathfrak{n}$. Thus H is topologically isomorphic to $C \times H/C$ (see [6], 24.45(b)). Apply Theorem 12.1 to C. The new product measure on $C \times H/C$ gives the desired extension.

A well known structure theorem for locally compact compactly generated Abelian groups states that every such group is topologically isomorphic with $R^a \times Z^b \times F$ (here R=real numbers, Z=integers) for some non-negative integers a and b and some compact Abelian group F. Thus if such a group G is connected it follows that $G \cong R^a \times F$ where F is a compact connected group.

12.3. Corollary. Let H be a locally compact connected compactly generated Abelian group. Suppose H has a compact subgroup J satisfying $w(H)=w(J)=\mathfrak{n} \geqslant \aleph_0$. Then there is a translation and inversion invariant extension of Haar measure on H of character $2^{2^{\mathfrak{n}}}$.

Finally if G is a locally compact locally connected compactly generated Abelian group it follows that $G \cong R^a \times Z^b \times F$ for some non-negative integers a and b and some locally connected compact Abelian group F. Thus:

12.4. Corollary. Let H be a locally compact locally connected compactly generated Abelian group. Suppose H has a compact subgroup J satisfying $w(H) = w(J) = \mathfrak{n}$. Then there is a translation and inversion invariant extension of Haar measure on H of character $2^{2^{\mathfrak{n}}}$.

Note. Since this work was completed, HEWITT and ROSS [7] have generalized and simplified Theorem 12.1 and Corollary 12.4. Their theorem implies Theorem 12.1 for all compact Abelian groups, and uses our Theorem 5.6, Theorem 8.6 and Lemma 8.8. Parts of their proof are similar to ours.

State University of New York at Buffalo.

REFERENCES

1. COMFORT, W. W. and K. A. ROSS, Pseudocompactness and uniform continuity in topological groups, to appear in Pacific J. Math.
2. ———— and ————, Topologies induced by groups of characters, to appear in Fundamenta Math.
3. DOOB, J. L., Stochastic processes depending on a continuous parameter, Trans. Amer. Math. Soc. **42**, 107–140 (1937).
4. HALMOS, P. R., Measure theory, New York, N.Y., D. Van Nostrand Co., 1950.
5. HARTMAN, S. and A. HULANICKI, Les ensembles denses dans les groupes topologiques, Coll. Math. **6**, 187–191 (1958).
6. HEWITT, E. and K. A. ROSS, Abstract harmonic analysis, Vol. I, Heidelberg, Springer-Verlag, 1963.
7. ———— and ————, Extensions of Haar measure and of harmonic analysis for locally compact Abelian groups, to appear in Math. Annalen.
8. HULANICKI, A., Algebraic characterization of Abelian divisible groups which admit compact topologies, Fund. Math. **44**, 192–197 (1957).
9. ————, Algebraic structure of compact Abelian groups, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys. **6**, 71–73 (1958).
10. ————, On subsets of full outer measure in products of measure spaces, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. Phys. **7**, 331–335 (1959).
11. KAKUTANI, S., On cardinal numbers related with a compact Abelian group. Proc. Imp. Acad. Tokyo **19**, 366–372 (1943).
12. ———— and K. KODAIRA, A non-separable translation invariant extension of the Lebesque measure space, Ann. of Math. **52**, 574–579 (1950).
13. ———— and J. C. OXTOBY, Construction of a non-separable invariant extension of the Lebesque measure space. Ann. of Math. (2) **52**, 580–590 (1950).
14. KELLEY, J. L., General topology, New York, D. Van Nostrand Co., 1955.
15. PONTRYAGIN, L. S., Continuous groups, 2nd Edition, Gostehizdat, Moscow, 1954.
16. RUDIN, W., Fourier analysis on groups, New York, Interscience Publishers, 1962.
17. VILENKIN, N. YA., On the dyadicity of the group space of bicompact commutative groups, Uspehi Mat. Nauk, N.S. **13**, vyp. 6 (84), 79–80 (1958).